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On the quest for positivity in operator algebras

C.J. Read

Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England, United Kingdom

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ABSTRACT

We show that in every nonzero operator algebra with a contractive approximate identity (or c.a.i.), there is a nonzero operator T such that $\|I-T\| \leqslant 1$. In fact, there is a c.a.i. consisting of operators T with $\|I-2T\| \leqslant 1$. So, the numerical range of the elements of our contractive approximate identity is contained in the closed disk center $\frac{1}{2}$ and radius $\frac{1}{2}$. This is the necessarily weakened form of the result for C^*-algebras, where there is always a contractive approximate identity consisting of operators with $0 \leqslant T \leqslant 1$ – the numerical range is contained in the real interval $[0,1]$. So, if an operator algebra has a c.a.i., it must have operators with a "certain amount" of positivity.

1. Introduction

For us, an *operator algebra* is a norm-closed subalgebra of the algebra $B(H)$ of all bounded linear operators on some Hilbert space H. Our main theorem is as follows:

Theorem 1.1. *Let $\mathscr{A} \subset B(H)$ be an operator algebra with given contractive approximate identity $(T_\alpha)_{\alpha\in A}$. Let Γ denote the closed convex hull of the operators T_α. Then for each $\varepsilon \in (0,1)$, there is a contractive approximate identity for $\mathscr{A}$ consisting of operators $T \in \Gamma$ satisfying $\|I-(2-\varepsilon)T\| \leqslant 1$. In particular, $\mathscr{A}$ has a contractive approximate identity consisting of operators T satisfying $\|I-2T\| \leqslant 1$.*

This result solves problems which have been studied by various authors [4,1,2]. D.P. Blecher and this author have written a paper rather longer than this one [3], giving further results obtainable from the result given in this paper. From the point of view of this author, Theorem 1.1 gives a really nice way of using the extra structure of an operator algebra (which a general Banach algebra need not have) to say something beautiful and useful about approximate identities. An operator algebra does not reveal its structure as easily as a C^*-algebra in this area. Simple examples show that an operator algebra need not have an approximate identity; if it does have an a.i., the algebra need not have any bounded a.i., and if it has a b.a.i. the algebra need not have any contractive a.i. Even if it has a b.a.i., an operator algebra need not contain any Hermitian element, nor even any nonzero element with $\|I-T\| \leqslant 1$ (it can be shown that an operator algebra has an essentially unique unitization, so this last question is independent of the choice of representation). Here we show, however, that if an operator algebra has a c.a.i., then it has a c.a.i. consisting of operators T with $\|I-2T\| \leqslant 1$. The method of proof is to show, for each $\varepsilon \in (0,1)$, that there is a c.a.i. (T_α) satisfying $\|I-(2-\varepsilon)T_\alpha\| \leqslant 1$ and every $T_\alpha \in \Gamma$. If $\|I-(2-\varepsilon)T_\alpha\| \leqslant 1$ then the operator $T'_\alpha = (1-\varepsilon/2)T_\alpha$ satisfies $\|I-2T'_\alpha\| \leqslant 1$, and the collection of all these (for all ε and all α) is, when suitably directed, a c.a.i. whose operators satisfy $\|I-2T\| \leqslant 1$.

E-mail address: read@maths.leeds.ac.uk.

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In the sequel [3], we begin by discussing open and closed projections, l-ideals, r-ideals and hereditary subalgebras (HSAs). Immediate corollaries of Theorem 1.1 tell us that every r-ideal of an operator algebra $\mathscr{A}$ has a left c.a.i. (e_t) with $\|1-2e_t\| \leqslant 1$, every l-ideal of $\mathscr{A}$ has a right c.a.i. (e_t) with $\|1-2e_t\| \leqslant 1$, and every HSA of $\mathscr{A}$ has a c.a.i. (e_t) with $\|1-2e_t\| \leqslant 1$. Furthermore if $x \in \mathscr{A}$ itself satisfies $\|1-x\| \leqslant 1$ then the closed subalgebra $oa(x) \subset \mathscr{A}$ generated by x has a c.a.i. – a hint of a converse to Theorem 1.1. If $\mathscr{A}$ is separable and has a c.a.i., it always has a sequential c.a.i. of form $e_n = x^{1/n}$ for suitable $x \in \mathscr{A}$ with $\|1-x\| \leqslant 1$ and suitable choices of the nth roots $x^{1/n}$.

We then discuss p-projections and peak projections, which are natural generalizations to the noncommutative setting of the p-sets and peak sets of the theory of function spaces. Theorem 1.1 is what is needed to prove the noncommutative version of Glicksberg's fundamental result: if $\mathscr{A}$ is a unital operator algebra, then every closed projection $q \in \mathscr{A}^{**}$ is a p-projection, and indeed is a strong limit of a decreasing net of peak projections for $\mathscr{A}$. The main unsolved question in the thesis of D.M. Hay [5] was whether this might be true; the result puts the theory of "noncommutative peak sets" on a much firmer foundation. A key observation in [2] was that the "noncommutative Glicksberg theorem" would follow if every algebra $\mathscr{A}$ as in Theorem 1.1 had a b.a.i. consisting of operators T with $\|I-T\| \leqslant 1$. In Theorem 1.1, we in fact do a little better than that.

Our sequel [3] concludes by giving a definition of "operator complete positivity" which generalizes the notion of complete positivity of maps between C^*-algebras, and again uses the key equation $\|I-T\| \leqslant 1$. Writing $\mathscr{F}_{\mathscr{A}}$ for the set of elements $x \in \mathscr{A}$ with $\|1-x\| \leqslant 1$, the map $T : \mathscr{A} \to \mathscr{B}$ is operator completely positive (or OCP) if there is a constant $C > 0$ such that $T_n(\mathscr{F}_{M_n(\mathscr{A})}) \subset C \cdot \mathscr{F}_{M_n(\mathscr{B})}$ for all $n \in \mathbb{N}$ (where T_n is the tensor product of T with the $n \times n$ identity matrix). This definition looks quite sensible, and some promising results are given.

2. The operators T_i and associated projections

In this paper, our sole objective is to prove the key Theorem 1.1. We begin by writing Γ for the closed convex hull of the given c.a.i. Γ is a subset of the unit ball of $\mathscr{A}$. To find, for a given $\varepsilon \in (0,1)$, the contractive approximate identity whose existence is asserted by the theorem, it is enough that for each $\zeta \in (0,1)$ and each finite sequence $S_1, \dots, S_n \in \mathscr{A}$ with $\|S_r\| = 1$, we find a $T \in \Gamma$ with $\|TS_r - S_r\| \vee \|S_rT - S_r\| \leqslant \zeta$, $r = 1, \dots, n$, and $\|I - (2-\varepsilon)T\| \leqslant 1$. Since the statements become stronger as ε decreases and also as ζ decreases, we may as well combine both variables by assuming $\zeta \in (0, \frac{1}{2})$ and $\varepsilon = \frac{2\zeta}{1+\zeta}$ (so $2 - \varepsilon = \frac{2}{1+\zeta}$).

So to prove the theorem, for a given $\zeta \in (0, \frac{1}{2})$ and given $S_1, \dots, S_n \in \mathscr{A}$, we seek a $T \in \Gamma$ such that $\|TS_r - S_r\| \vee \|S_rT - S_r\| \leqslant \zeta$, and $\|I - \frac{2T}{1+\zeta}\| \leqslant 1$.

Definition 2.1. Let us pick a sequence $(T_i)_{i=0}^{\infty} \in \Gamma$ as "witnesses" to the fact that Γ contains a c.a.i. for an algebra containing the operators S_r. We demand that

(i) $\|T_iS_r - S_r\| \vee \|S_rT_i - S_r\| \leqslant \zeta$ for all i, r;
(ii) for $j > i$, $\|T_jT_i - T_i\| \vee \|T_iT_j - T_i\| \leqslant \delta_j = 1/D_j$,

where $\mathbf{D} = (D_j)_{j=0}^{\infty}$ is a "rapidly increasing sequence" of positive integers, satisfying growth conditions to be specified later.

Let $dP^{(i)}$ (respectively, $dQ^{(i)}$) be the spectral measures for $T_i^*T_i$ (respectively, $T_iT_i^*$), so we have

$$T_i^*T_i = \int_{\lambda \in [0,1]} \lambda \cdot dP^{(i)}(\lambda), \tag{1}$$

$$T_iT_i^* = \int_{\lambda \in [0,1]} \lambda \cdot dQ^{(i)}(\lambda). \tag{2}$$

Useful integrals we shall perform include:

$$P_{i,\eta} = \int_{\lambda \in (\eta,1]} dP^{(i)}(\lambda), \tag{3}$$

$$Q_{i,\eta} = \int_{\lambda \in (\eta,1]} dQ^{(i)}(\lambda). \tag{4}$$

Any $\eta \in (0,1)$ can be used, but a special choice is $\eta = \eta_i = \delta_i^{1/2}$. We define

$$\eta_i = \delta_i^{1/2}, \qquad P_i = P_{i,\eta_i}, \qquad Q_i = Q_{i,\eta_i}. \tag{5}$$

Now if $x \in \ker P_{i,\eta}$ then $\langle T_i^* T_i x, x\rangle \leqslant \eta\|x\|^2$, that is, $\|Tx\| \leqslant \eta^{1/2}\|x\|$. So,

$$\left\|T_i(I - P_{i,\eta})\right\| \leqslant \eta^{1/2}, \tag{6}$$

and likewise

$$\left\|T_i^*(I - Q_{i,\eta})\right\| \leqslant \eta^{1/2}. \tag{7}$$

So,

$$\|T_i - Q_{i,\eta} T_i P_{i,\eta}\| \leqslant 2\eta^{1/2}. \tag{8}$$

3. Basic properties of T_i, $Q_{i,\eta}$ and $P_{i,\eta}$

Lemma 3.1. *For $n > i$ and $\eta \in (0, 1)$ we have*

$$\|T_n Q_{i,\eta} - Q_{i,\eta}\| \leqslant \delta_n/\eta \tag{9}$$

and

$$\|P_{i,\eta} T_n - P_{i,\eta}\| \leqslant \delta_n/\eta. \tag{10}$$

Proof. If $x \in \operatorname{Im} Q_{i,\eta}$ with $\|x\| = 1$ we have $x = T_i T_i^* y$ with $\|y\| \leqslant 1/\eta$ (write $x = \int_{\lambda\in(\eta,1]} dQ^{(i)}(\lambda)[x]$ and consider $y = \int_{\lambda\in(\eta,1]} \lambda^{-1}\, dQ^{(i)}(\lambda)[x]$). So

$$\|T_n x - x\| = \left\|T_n T_i T_i^* y - T_i T_i^* y\right\| \leqslant \delta_n \left\|T_i^* y\right\| \quad \text{(from Definition 2.1)} \tag{11}$$

$$\leqslant \delta_n/\eta. \tag{12}$$

Hence we get (9). Similarly we can show $\|T_n^* P_{i,\eta} - P_{i,\eta}\| \leqslant \delta_n/\eta$, and that is equivalent to (10). $\square$

Lemma 3.2. *For $n > i$ and $\eta \in (\delta_n, 1)$ we have*

$$\|Q_{i,\eta} T_n - Q_{i,\eta}\| \leqslant 3(\delta_n/\eta)^{1/2}, \tag{13}$$

$$\|T_n P_{i,\eta} - P_{i,\eta}\| \leqslant 3(\delta_n/\eta)^{1/2}. \tag{14}$$

Proof. Let $x \in \operatorname{Im} Q_{i,\eta}$ with $\|x\| = 1$. Then

$$1 \geqslant \left\|T_n^* x\right\|^2 = \left\|Q_{i,\eta} T_n^* x\right\|^2 + \left\|(I - Q_{i,\eta}) T_n^* x\right\|^2 \geqslant (1 - \delta_n/\eta)^2 + \left\|(I - Q_{i,\eta}) T_n^* x\right\|^2 \tag{15}$$

by (9). Therefore

$$\left\|(I - Q_{i,\eta}) T_n^* Q_{i,\eta}\right\|^2 \leqslant 1 - (1 - \delta_n/\eta)^2 = 2\delta_n/\eta - \delta_n^2/\eta^2. \tag{16}$$

On the other hand, $\|Q_{i,\eta} T_n^* Q_{i,\eta} - Q_{i,\eta}\| \leqslant \delta_n/\eta$ by (9), so we have $\|T_n^* Q_{i,\eta} - Q_{i,\eta}\| \leqslant \|(I - Q_{i,\eta}) T_n^* Q_{i,\eta}\| + \|Q_{i,\eta} T_n^* Q_{i,\eta} - Q_{i,\eta}\| \leqslant (2\delta_n/\eta)^{1/2} + \delta_n/\eta \leqslant 3(\delta_n/\eta)^{1/2}$. That gives (13); (14) follows similarly. $\square$

Lemma 3.3. *For $n > i$ and $\eta \in (\delta_n, 1)$, $\eta' \in (0, 1)$, we have*

$$\|Q_{n,\eta'} Q_{i,\eta} - Q_{i,\eta}\| = \|Q_{i,\eta} Q_{n,\eta'} - Q_{i,\eta}\| \leqslant 3\left(\sqrt{\eta'} + \sqrt{\delta_n/\eta}\right)^{1/2}, \tag{17}$$

$$\|P_{n,\eta'} Q_{i,\eta} - Q_{i,\eta}\| = \|Q_{i,\eta} P_{n,\eta'} - Q_{i,\eta}\| \leqslant 2\left(\sqrt{\eta'} + \delta_n/\eta\right)^{1/2}, \tag{18}$$

$$\|Q_{n,\eta'} P_{i,\eta} - P_{i,\eta}\| = \|P_{i,\eta} Q_{n,\eta'} - P_{i,\eta}\| \leqslant 2\left(\sqrt{\eta'} + \delta_n/\eta\right)^{1/2}, \tag{19}$$

$$\|P_{n,\eta'} P_{i,\eta} - P_{i,\eta}\| = \|P_{i,\eta} P_{n,\eta'} - P_{i,\eta}\| \leqslant 3\left(\sqrt{\eta'} + \sqrt{\delta_n/\eta'}\right)^{1/2}. \tag{20}$$

Proof. If $x \in \ker Q_{n,\eta'}$ with $\|x\| \leqslant 1$ then $\|T_n^* x\| \leqslant \sqrt{\eta'}$ by (7). But for $y \in \operatorname{Im} Q_{i,\eta}$ with $\|y\| = 1$, (13) gives $\|T_n^* y - y\| \leqslant 3(\delta_n/\eta)^{1/2}$. Therefore

$$1 - 3(\delta_n/\eta)^{1/2} \leqslant \left\|T_n^* y\right\| \leqslant \left\|T_n^* Q_{n,\eta'} y\right\| + \left\|T_n^*(I - Q_{n,\eta'}) y\right\| \leqslant \|Q_{n,\eta'} y\| + \sqrt{\eta'}, \tag{21}$$

$$\|Q_{n,\eta'} y\| \geqslant 1 - 3(\delta_n/\eta)^{1/2} - \sqrt{\eta'}. \tag{22}$$

Since $Q_{n,\eta'}$ is an orthogonal projection, that implies

$$\begin{aligned}\big\|(I-Q_{n,\eta'})y\big\| &\leqslant \big(1-\big(1-3(\delta_n/\eta)^{1/2}-\sqrt{\eta'}\big)^2\big)^{1/2}\\ &\leqslant \big(6(\delta_n/\eta)^{1/2}+2\sqrt{\eta'}\big)^{1/2}\leqslant 3\big(\sqrt{\eta'}+\sqrt{\delta_n/\eta}\big)^{1/2},\end{aligned}\tag{23}$$

and hence we have (17). (20) is established similarly. To get (18) (and (19)), we note that for $x\in\ker P_{n,\eta'}$ with $\|x\|\leqslant 1$ we have $\|T_nx\|\leqslant\sqrt{\eta'}$ by (6); but if $y\in\operatorname{Im} Q_{i,\eta}$ with $\|y\|=1$, we have $\|T_ny-y\|\leqslant\delta_n/\eta$ by (9). Therefore

$$1-\delta_n/\eta\leqslant\|T_ny\|\leqslant\|T_nP_{n,\eta'}y\|+\big\|T_n(I-P_{n,\eta'})y\big\|\leqslant\|P_{n,\eta'}y\|+\sqrt{\eta'},\tag{24}$$

$$\|P_{n,\eta'}y\|\geqslant 1-\delta_n/\eta-\sqrt{\eta'};\tag{25}$$

and since $P_{n,\eta'}$ is an orthogonal projection, that implies

$$\big\|(I-P_{n,\eta'})y\big\|\leqslant\big(1-\big(1-\delta_n/\eta-\sqrt{\eta'}\big)^2\big)^{1/2}\leqslant\big(2\delta_n/\eta+2\sqrt{\eta'}\big)^{1/2};\tag{26}$$

which gives us (18) and, similarly, (19). $\square$

Lemma 3.4. *If the sequence D_n tends to infinity, the following is true: the subspaces $E=\bigcap_{n=0}^\infty\ker T_n$, $\bigcap_{n=0}^\infty\ker T_n^*$, $\bigcap_{n=0}^\infty\ker Q_n$ and $\bigcap_{n=0}^\infty\ker P_n$ are all equal.*

Proof. Suppose $T_nx=0$ for all n. Then (1) gives us $\int_0^1\lambda\,dP^{(n)}(\lambda)[x]=0$, in particular $P_n(x)=0$ for all n. But for every $i<n$, we know from (18) that

$$\|Q_iP_n-Q_i\|\leqslant 2\big(\delta_n^{1/4}+\delta_n/\sqrt{\delta_i}\big)^{1/2}\leqslant 3\delta_n^{1/8}$$

so given $D_n\to\infty$ we find that $Q_ix=\lim_{n\to\infty}Q_iP_nx=0$, and $x\in\bigcap_n\ker Q_n$. But then, (17) tells us that for each $n>i$ and each $\eta\in(\delta_n,1)$,

$$\|Q_{i,\eta}Q_n-Q_{i,\eta}\|\leqslant 3\big(\delta_n^{1/4}+(\delta_n/\eta)^{1/2}\big)^{1/2}.\tag{27}$$

So we conclude that $Q_{i,\eta}(x)=\lim_{n\to\infty}Q_{i,\eta}Q_n(x)=0$. Then (7) tells us that $T_i^*x=\lim_{\eta\to 0}T_i^*Q_{i,\eta}(x)=0$, so $x\in\bigcap_n\ker T_n^*$. This argument is reversible by swapping the roles of T and T^*, Q and P; so $x\in\bigcap\ker T_n^*$ implies $x\in\bigcap\ker Q_n$ implies $x\in\bigcap\ker P_n$ implies $x\in\bigcap\ker T_n$. Thus the lemma is proved. $\square$

4. The nearly orthogonal decomposition

It turns out that we can construct the orthogonal projection onto $E^\perp$, E being the subspace of Lemma 3.4, out of the projections we have already defined.

Definition 4.1. Let

$$\begin{aligned}\pi_n&=\prod_{i=0}^n(I-P_{n-i})=(I-P_n)(I-P_{n-1})(I-P_{n-2})\cdots(I-P_0),\\ \rho_n&=\prod_{i=0}^n(I-Q_{n-i})=(I-Q_n)(I-Q_{n-1})(I-Q_{n-2})\cdots(I-Q_0),\\ \bar P_n&=P_n\pi_{n-1},\qquad \bar Q_n=Q_n\rho_{n-1};\end{aligned}\tag{28}$$

where π_{-1} and ρ_{-1} are deemed to be the identity.

Lemma 4.2. *For every n we have*

$$\|\rho_n-I+Q_n\|\leqslant 5n\delta_n^{1/8},\tag{29}$$

$$\|\pi_n-I+P_n\|\leqslant 5n\delta_n^{1/8}.\tag{30}$$

Proof. We know from (27) that for $i<n$,

$$\|Q_iQ_n-Q_i\|\leqslant 3\big(\delta_n^{1/4}+\big(\delta_n/\delta_i^{1/2}\big)^{1/2}\big)^{1/2}\leqslant 5\delta_n^{1/8}.\tag{31}$$

So,

$$\big\|(I-Q_n)(I-Q_{n-1})-(I-Q_n)\big\|=\big\|(I-Q_n)Q_{n-1}\big\|\leqslant 5\delta_n^{1/8};$$

then

$$\begin{aligned}&\bigl\|(I-Q_n)(I-Q_{n-1})(I-Q_{n-2})-(I-Q_n)\bigr\|\\&\quad\leqslant 5\delta_n^{1/8}+\bigl\|(I-Q_n)(I-Q_{n-2})-(I-Q_{n-2})\bigr\|\leqslant 10\delta_n^{1/8};\end{aligned}$$

and so on, for a grand total of $\|\rho_n-(I-Q_n)\|\leqslant 5n\delta_n^{1/8}$ as required. Similarly, we have $\|\pi_n-(I-P_n)\|\leqslant 5n\delta_n^{1/8}$ also. $\square$

Lemma 4.3. *Suppose the underlying sequence* **D** *satisfies the growth condition $n\delta_n^{1/8}\to 0$. Then for all $x\in H$, the sums $\sum_{n=0}^{\infty}\bar{Q}_n x$ and $\sum_{n=0}^{\infty}\bar{P}_n x$ both converge to $\bar{Q}x$, where $\bar{Q}$ is the orthogonal projection onto $E^{\perp}$, and E is as in Lemma* 3.4.

Proof. An elementary induction shows that for all $n\in\mathbb{N}_0$ we have

$$I-\sum_{i=0}^{n}\bar{Q}_i=\rho_n,\qquad I-\sum_{i=0}^{n}\bar{P}_i=\pi_n. \tag{32}$$

So suppose $x\in\ker Q_i$ for all i. Plainly $\bar{Q}_i x=0$ for all i, and $\sum_{i=0}^{\infty}\bar{Q}_i x=0$. On the other hand, if $\|x\|=1$ and $x\in\operatorname{Im}Q_j$ for some j, we have

$$\Biggl\|x-\sum_{i=0}^{n}\bar{Q}_i x\Biggr\|=\|\rho_n x\|\leqslant\bigl\|(I-Q_n)x\bigr\|+5n\delta_n^{1/8}, \tag{33}$$

by (29). Since $x\in\operatorname{Im}Q_j$, $(I-Q_n)x\to 0$ by (31), so given $n\delta_n^{1/8}\to 0$ we know $\sum_{n=0}^{\infty}\bar{Q}_n x=x$, and the same is true if x is a finite sum of vectors $x_i\in\operatorname{Im}Q_i$. Now it is clear that $\|\pi_n\|,\|\rho_n\|\leqslant 1$; so (32) tells us that the partial sums $\sum_{i=0}^{n}\bar{Q}_i x$ have norm at most $2\|x\|$ for any n and x.

In particular, the seminorm $p(x)=\limsup_n\|x-\sum_{i=0}^{n}\bar{Q}_i x\|$ is at most $3\|x\|$ for any x and is therefore continuous with respect to the usual norm on $\mathscr{A}$. So $p^{-1}(\{0\})=\{x\colon \sum\bar{Q}_i x=x\}$ is closed; it contains $\overline{\sum_{i=0}^{\infty}\operatorname{Im}Q_i}$. But $x\perp\overline{\sum_{i=0}^{\infty}\operatorname{Im}Q_i}$ if and only if $x\in\bigcap\ker Q_i=E$, so the sum $\sum\bar{Q}_i x$ is equal to zero for $x\in E$, and to x for $x\in E^{\perp}$. Therefore, $\sum\bar{Q}_i x=\bar{Q}x$ as claimed, and likewise $\sum\bar{P}_i x=\bar{Q}x$ also. $\square$

5. Defining the alternative norm, and the operator T

Definition 5.1. Let $M>100$ be chosen (once and for all, and independent of the sequence **D**). Define $a_n=\frac{M}{(n+M)(n+M+1)}$. Note that $\sum_{n=0}^{\infty}a_n=1$, and define $T=\sum_{i=0}^{\infty}a_i T_i$, $s_n=\sum_{m=n}^{\infty}a_m=\frac{M}{n+M}$. Note that for all n,

$$\frac{s_{n+1}}{a_n}\geqslant M;\qquad \frac{s_{n+2}}{a_n}\geqslant\frac{M(M+1)}{M+2}. \tag{34}$$

We define what we claim is a seminorm on H as follows:

$$|||x|||=\Biggl(\sum_{n=0}^{\infty}s_{n+1}\|\bar{Q}_n x\|^2\Biggr)^{1/2}. \tag{35}$$

We shall show that $|||x|||<\infty$ for all $x\in H$. Once this is established, we shall find that the seminorm, when restricted to $E^{\perp}$, is an alternative norm on $E^{\perp}$. This is because for $x\in E^{\perp}$ we have $x=\sum_{n=0}^{\infty}\bar{Q}_n x$ by Lemma 4.3. We will also prove convergence of the sum

$$\tau(x)=\sum_{n=0}^{\infty}\bar{Q}_n^2 x. \tag{36}$$

Of course, τ would be the projection $\bar{Q}$ if $\bar{Q}_n$ were a projection, by Lemma 4.3; but in general this is not so, and the magnitude of the error is measured as follows:

Lemma 5.2. *Given growth conditions on the sequence* **D**, *the following is true*: *for all $n\in\mathbb{N}_0$, $\|\bar{Q}_n-\bar{Q}_n^2\|\leqslant 15n\delta_{n-1}^{1/8}$ (if $n>0$), or 0 (if $n=0$).*

Proof. If $n=0$ then $\bar{Q}_n^2=Q_0^2=Q_0=\bar{Q}_n$. If $n>0$, (29) tells us $\bar{Q}_n=Q_n\rho_{n-1}=Q_n(I-Q_{n-1}+h)$, with $\|h\|\leqslant 5(n-1)\delta_{n-1}^{1/8}$. So, since $\|Q_n\|,\|I-Q_{n-1}\|\leqslant 1$, we have $\|\bar{Q}_n-Q_n(I-Q_{n-1})\|\leqslant\|h\|$ and $\|\bar{Q}_n^2-(Q_n(I-Q_{n-1}))^2\|\leqslant 2\|h\|+\|h\|^2$. So, $Q_n(I-Q_{n-1})-(Q_n(I-Q_{n-1}))^2=Q_n(I-Q_{n-1})-Q_n(I-Q_{n-1})Q_n(I-Q_{n-1})=Q_nQ_{n-1}Q_n(I-Q_{n-1})=Q_n[Q_{n-1},Q_n](I-Q_{n-1})$

since Q_n is a projection; and the commutator $[Q_{n-1}, Q_n]$ has norm at most $\|Q_{n-1}Q_n - Q_{n-1}\| + \|Q_nQ_{n-1} - Q_{n-1}\| \leqslant 10\delta_n^{1/8}$ by (31) (obviously $\|Q_{n-1}Q_n - Q_{n-1}\| = \|Q_nQ_{n-1} - Q_{n-1}\|$ since the second operator is the hermitian conjugate of the first). So

$$\|\bar{Q}_n - \bar{Q}_n^2\| \leqslant 3\|h\| + \|h\|^2 + 10\delta_n^{1/8} \leqslant 15(n-1)\delta_{n-1}^{1/8} + 25(n-1)^2\delta_{n-1}^{1/4} + 10\delta_n^{1/8}; \tag{37}$$

if $\mathbf{D}$ increases sufficiently rapidly (so that $\delta_n = 1/D_n$ decreases sufficiently rapidly), we can be sure, then, that $\|\bar{Q}_n - \bar{Q}_n^2\| \leqslant 15n\delta_{n-1}^{1/8}$. $\square$

Corollary 5.3. *Let $\eta_1 \in (0,1)$ be given. If $\mathbf{D}$ increases sufficiently rapidly, the following is true: the sum $\tau(x)$ in (36) converges for every $x \in H$, and for $x \in E^\perp$ we have*

$$\|x - \tau x\| \leqslant \eta_1\|x\|. \tag{38}$$

Proof. If $x \in E$ obviously $\tau x = 0$. If $x \in E^\perp$, we know $x = \sum_n \bar{Q}_n x$, so from (36) we have $\|x - \tau x\| \leqslant \sum_{n=0}^\infty \|(\bar{Q}_n - \bar{Q}_n^2)x\| \leqslant (\sum_{n=1}^\infty 15n\delta_{n-1}^{1/8}) \cdot \|x\|$, by Lemma 5.2. If $\mathbf{D}$ grows suitably fast, we will indeed have $\|x - \tau x\| \leqslant \eta_1\|x\|$. $\square$

Lemma 5.4. *Given growth conditions on $\mathbf{D}$, the following statements are true: for all $n \geqslant 1$,*

$$\|\bar{Q}_n - Q_n + Q_{n-1}\| \leqslant 5n\delta_{n-1}^{1/8}; \tag{39}$$

and for all $n > m$,

$$\|\bar{Q}_n|_{\operatorname{Im} Q_m}\| \vee \|\bar{Q}_n^*|_{\operatorname{Im} Q_m}\| \leqslant 5(n+2)\delta_{n-1}^{1/8}. \tag{40}$$

In particular, for $n > m$

$$\|\bar{Q}_n\bar{Q}_m\| \vee \|\bar{Q}_n^*\bar{Q}_m\| \leqslant 5(n+2)\delta_{n-1}^{1/8}. \tag{41}$$

Also for $n < m$,

$$\|\bar{Q}_n\bar{Q}_m\| \leqslant 5(m + 2^{m+1})\delta_{m-1}^{1/8}. \tag{42}$$

Proof. By (29), $\bar{Q}_n = Q_n\rho_{n-1} = Q_n(I - Q_{n-1} + h)$, $\|h\| \leqslant 5(n-1)\delta_{n-1}^{1/8}$. By (31), $\|Q_nQ_{n-1} - Q_{n-1}\| \leqslant 5\delta_n^{1/8} < 5\delta_{n-1}^{1/8}$. So, $\|\bar{Q}_n - Q_n + Q_{n-1}\| \leqslant 5n\delta_{n-1}^{1/8}$ as claimed in (39). Since Q_n, Q_{n-1} are selfadjoint, that means $\|\bar{Q}_n^* - Q_n + Q_{n-1}\| \leqslant 5n\delta_{n-1}^{1/8}$ also. But for $n > m$,

$$\begin{aligned}\|(Q_n - Q_{n-1})Q_m\| &\leqslant \|Q_nQ_m - Q_m\| + \|Q_{n-1}Q_m - Q_m\| \\ &\leqslant \begin{cases} 5\delta_n^{1/8} + 5\delta_{n-1}^{1/8}, & \text{if } n > m+1, \\ 5\delta_n^{1/8}, & \text{if } n = m+1, \end{cases}\end{aligned} \tag{43}$$

by two (or one) applications of (31). So if $S = \bar{Q}_n$ or $\bar{Q}_n^*$, we have $\|S|_{\operatorname{Im} Q_m}\| \leqslant 5n\delta_{n-1}^{1/8} + \|(Q_n - Q_{n-1})Q_m\| \leqslant 5(n+2)\delta_{n-1}^{1/8}$, establishing (40). (41) now follows because $\bar{Q}_m = Q_m\bar{Q}_m$ and $\|\bar{Q}_m\| \leqslant 1$. Finally we note that $\bar{Q}_n = Q_n\rho_{n-1}$ is a sum of 2^n monomials $Q_nQ_{r_1}Q_{r_2}\cdots Q_{r_k}$ with $k \geqslant 0$ and the $r_j < n$; so for $m > n$, using (39) we have $\|\bar{Q}_n\bar{Q}_m\| \leqslant 5m\delta_{m-1}^{1/8} + \|\bar{Q}_n(Q_m - Q_{m-1})\| \leqslant 5m\delta_{m-1}^{1/8} + 2^n \cdot \max\{\|Q_r(Q_m - Q_{m-1})\|\colon r \leqslant n\} \leqslant 5(m + 2^{m+1})\delta_{m-1}^{1/8}$ by (43). Thus we have (42). $\square$

Lemma 5.5. *Let $(\alpha_n)_{n=0}^\infty$ be a decreasing sequence of positive real numbers tending to zero. Given appropriate growth conditions on $\mathbf{D}$, the following will be true: for all $x \in H$ and $k \in \mathbb{N}_0$,*

$$\sum_{n=k}^\infty \|(\bar{Q}_n - \bar{Q}_n^2)x\| \leqslant \sum_{n=0}^\infty \alpha_{n\vee k}\|\bar{Q}_n x\|. \tag{44}$$

Proof. For $\bar{Q}x = \sum_m \bar{Q}_m x$, $\bar{Q}_n x = \bar{Q}_n\bar{Q}x = \sum_{m=0}^\infty \bar{Q}_n\bar{Q}_m x$, so $(\bar{Q}_n - \bar{Q}_n^2)x = \sum_{m\in\mathbb{N}_0, m\neq n} \bar{Q}_n\bar{Q}_m x$. Applying (40) to the part of the sum with $m < n$, we have $\|(\bar{Q}_n - \bar{Q}_n^2)x\| \leqslant \sum_{m=0}^{n-1} 5(n+2)\delta_{n-1}^{1/8}\|\bar{Q}_m x\| + \sum_{m=n+1}^\infty \|\bar{Q}_n\bar{Q}_m x\|$. We estimate the second sum by a sum of $\|\bar{Q}_n\bar{Q}_m^2 x\|$ plus a sum of $\|\bar{Q}_n(\bar{Q}_m - \bar{Q}_m^2)x\|$, thus: $\sum_{m=n+1}^\infty \|\bar{Q}_n\bar{Q}_m x\| \leqslant \sum_{m=n+1}^\infty \|\bar{Q}_n\bar{Q}_m\| \cdot \|\bar{Q}_m x\| + \sum_{m=n+1}^\infty \|\bar{Q}_n(\bar{Q}_m^2 - \bar{Q}_m)x\| \leqslant \sum_{m=n+1}^\infty 5(m + 2^m)\delta_{m-1}^{1/8}\|\bar{Q}_m x\| + \sum_{m=n+1}^\infty \|\bar{Q}_n(\bar{Q}_m^2 - \bar{Q}_m)x\|$, using (42). So, if we define $u_n = \|(\bar{Q}_n - \bar{Q}_n^2)x\|$ and $\varepsilon_m = 5(m + 2^m)\delta_{m-1}^{1/8}$, we have $u_n \leqslant \sum_{m\in\mathbb{N}_0, m\neq n} \varepsilon_{m\vee n}\|\bar{Q}_m x\| + \sum_{m=n+1}^\infty u_m$. Writing $U_m = (m+4)!u_m$, we have

$$\sum_{n=N}^{\infty} U_n \leqslant \sum_{n=N}^{\infty} \sum_{m\in\mathbb{N}_0, m\neq n} (m+4)!\varepsilon_{m\vee n} \|\bar{Q}_m x\| + \sum_{n=N}^{\infty} \sum_{m=n+1}^{\infty} \frac{(n+4)!}{(m+4)!} U_m. \tag{45}$$

On the right, the total coefficient of $\|\bar{Q}_m x\|$ is at most

$$\sum_{n\geqslant N, n\neq m} (n+4)!\varepsilon_{m\vee n} \leqslant \begin{cases} (m+5)!\varepsilon_m + \sum_{n=m+1}^{\infty}(n+4)!\varepsilon_n, & \text{if } m \geqslant N\vee 1 \\ \sum_{n=N\vee 1}^{\infty}(n+4)!\varepsilon_n, & \text{if } m < N \text{ or } m = 0 \end{cases} \leqslant \beta_{m\vee N},$$

where $\beta_m = \sum_{n=m\vee 1}^{\infty}(n+5)!\varepsilon_n$. The total coefficient of U_m on the right of (45) is zero if $m \leqslant N$, or $\frac{1}{(m+4)!}[(N+4)! + (N+5)! + \cdots + (m+3)!] \leqslant 2/(m+4)$, if $m > N$. So we have the inequality

$$\sum_{n=N}^{\infty} U_n \leqslant \sum_{m=0}^{\infty} \beta_{m\vee n} \|\bar{Q}_m x\| + \sum_{m=N+1}^{\infty} \frac{2}{m+4} U_m \leqslant \sum_{m=0}^{\infty} \beta_{m\vee n} \|\bar{Q}_m x\| + \frac{1}{2} \sum_{m=N+1}^{\infty} U_m. \tag{46}$$

The crude estimate $\|\bar{Q}_n^2 - \bar{Q}_n\| \leqslant 15n\delta_{n-1}^{1/8}$ from Lemma 5.2 can be used to tell us $\sum_{n=1}^{\infty} U_n \leqslant \|x\| \cdot \sum_{n=1}^{\infty}(n+4)! \cdot 15n\delta_{n-1}^{1/8}$, and given a growth condition on **D**, this is certainly finite. So it is legitimate to subtract $\frac{1}{2}\sum_{n=N}^{\infty} U_n$ from both sides of (46) and multiply by 2, obtaining

$$\sum_{n=N}^{\infty} U_n \leqslant 2 \sum_{m=0}^{\infty} \beta_{m\vee N} \|\bar{Q}_m x\|. \tag{47}$$

Throwing away the factors $(n+4)!$, we have $\sum_{n=N}^{\infty} u_n \leqslant 2\sum_{m=0}^{\infty} \beta_{m\vee N}\|\bar{Q}_m x\|$, and given growth conditions on **D**, we will have $\beta_r \leqslant \frac{1}{2}\alpha_r$ for all r (the condition we need is $\sum_{m=r}^{\infty}(m+5)! \cdot 5(m+2^m)\delta_{m-1}^{1/8} \leqslant \frac{1}{2}\alpha_r$ for all $r \in \mathbb{N}$; suitable growth conditions on **D** will certainly give it). Then we have (44). □

Lemma 5.6. *Let $\eta_2 \in (0,1)$ be given. If* **D** *increases sufficiently rapidly, the following is true: for all $x, y \in H$ we have*

$$\sum_{n,m\in\mathbb{N}_0; n\neq m} \left|\langle \bar{Q}_n x, \bar{Q}_m y\rangle\right| \leqslant \eta_2 \|x\| \cdot \|y\|. \tag{48}$$

In particular, for $x \in E^{\perp}$ we have

$$(1-\eta_2)\|x\|^2 \leqslant \sum_{n=0}^{\infty} \|\bar{Q}_n x\|^2 \leqslant (1+\eta_2)\|x\|^2. \tag{49}$$

Proof. For $n \neq m$ we have $\|\langle \bar{Q}_n x, \bar{Q}_m y\rangle\| \leqslant \|x\| \cdot \|y\| \cdot \|\bar{Q}_n^* \bar{Q}_m\| = \|x\| \cdot \|y\| \cdot \|\bar{Q}_m^* \bar{Q}_n\|$. By Lemma 5.4 (with n and m swapped if $m > n$) the norm of the operator is at most $5(n\vee m+2)\delta_{n\vee m-1}^{1/8}$, so

$$\left\|\langle \bar{Q}_n x, \bar{Q}_m y\rangle\right\| \leqslant \|x\| \cdot \|y\| \cdot \sum_{n\neq m; n,m\in\mathbb{N}_0} 5(n\vee m+2)\delta_{n\vee m-1}^{1/8}. \tag{50}$$

The sum is $\sum_{r=1}^{\infty} 5r(r+2)\delta_r^{1/8} \leqslant \eta_2$, given a growth condition on **D**. That proves (48); now if $x \in E^{\perp}$ we have $x = \sum_{n=0}^{\infty} \bar{Q}_n x$ so $\|x\|^2 = \sum_{n,m=0}^{\infty}\langle \bar{Q}_n x, \bar{Q}_m x\rangle = \sum_{n=0}^{\infty}\|\bar{Q}_n x\|^2 + \sum_{n,m\in\mathbb{N}_0; n\neq m}\langle \bar{Q}_n x, \bar{Q}_m y\rangle$. (49) then follows immediately from (50). □

Corollary 5.7. *Let $\eta_2 \in (0,1)$ be given. Given growth conditions on* **D**, *the norm $|||x|||$ exists for every $x \in H$, and is at most $\sqrt{1+\eta_2}\|x\|$.*

Proof. Putting the given value η_2 in Lemma 5.6, we have

$$|||x|||^2 = \sum_{n=0}^{\infty} s_{n+1}\|\bar{Q}_n x\|^2 \leqslant \sum_{n=0}^{\infty}\|\bar{Q}_n x\|^2 \leqslant (1+\eta_2)\|x\|^2, \tag{51}$$

given suitable growth conditions on **D**. □

6. Estimates concerning the operator τ

Lemma 6.1. *Let $\eta_3 \in (0,1)$ be given. If* **D** *increases sufficiently rapidly, the following are true*:

$$\|I-\tau^{-1}: E^\perp \to E^\perp\| \leqslant \eta_3, \tag{52}$$

$$|||I-\tau^{-1}: E^\perp \to E^\perp||| \leqslant \eta_3, \tag{53}$$

$$\|I-\tau^{-1}: (E^\perp, |||\cdot|||) \to (E^\perp, \|\cdot\|)\| \leqslant \eta_3, \tag{54}$$

$$\|I-\tau: (E^\perp, |||\cdot|||) \to (E^\perp, \|\cdot\|)\| \leqslant \eta_3. \tag{55}$$

Furthermore, for all $y \in E^\perp$ we have

$$\sum_{n=0}^{\infty} \|(\bar{Q}_n - \bar{Q}_n^2)y\| \leqslant \eta_3 |||y|||. \tag{56}$$

Note that (up to a small constant factor $\sqrt{1+\eta_2}$) Corollary 5.7 tells us that (54) is the strongest of the first three statements.

Proof of Lemma 6.1. If (38) is satisfied with $\eta_1 < 1$, certainly an inverse map $\tau^{-1}: E^\perp \to E^\perp$ exists. For a general $y \in E^\perp$, we have $(I-\tau)y = \sum_{n=1}^{\infty}(\bar{Q}_n - \bar{Q}_n^2)y$ so Lemma 5.5 tells us that

$$\|(I-\tau)y\| \leqslant \sum_{n=1}^{\infty} \|(\bar{Q}_n - \bar{Q}_n^2)y\| \leqslant \sum_{n=0}^{\infty} \alpha_n \|\bar{Q}_n y\| \leqslant \left(\sum_{n=0}^{\infty} \alpha_n\right)^{1/2} \left(\sum_{n=0}^{\infty} \alpha_n \|\bar{Q}_n y\|^2\right)^{1/2}$$

for any sequence $(\alpha_n) > 0$ we care to name, provided **D** satisfies suitable growth conditions. If the sequence (α_n) is suitably chosen, that certainly implies that

$$\|(I-\tau)y\| \leqslant \sum_{n=1}^{\infty} \|(\bar{Q}_n - \bar{Q}_n^2)y\| \leqslant \eta_3 |||y|||/4 \tag{57}$$

for all $y \in H$, giving us (55) and (56). And if we arrange that (38) is satisfied with $\eta_1 = 1/2$, for $x \in H$ we have $(I-\tau^{-1})x = (I-(I-(I-\tau))^{-1})x = \sum_{n=1}^{\infty}(I-\tau)^n x$, so $\|(I-\tau^{-1}): (E^\perp, |||\cdot|||) \to (E^\perp, \|\cdot\|)\| \leqslant \sum_{n=1}^{\infty} \|I-\tau\|^{n-1} \|I-\tau: (E^\perp, |||\cdot|||) \to (E^\perp, \|\cdot\|)\| \leqslant \eta_3/(4(1-\eta_1)) \leqslant \eta_3/2$. That establishes (54); then by Corollary 5.7, given suitable growth conditions we have $\|I-\tau^{-1}\| \leqslant \|I: (E^\perp, \|\cdot\|) \to (E^\perp, |||\cdot|||)\| \cdot \|(I-\tau^{-1}): (E^\perp, |||\cdot|||) \to (E^\perp, \|\cdot\|)\| \leqslant \eta_3\sqrt{1+\eta_2}/2$ and $|||I-\tau^{-1}||| \leqslant \|(I-\tau^{-1}): (E^\perp, |||\cdot|||) \to (E^\perp, \|\cdot\|)\| \times \|I: (E^\perp, \|\cdot\|) \to (E^\perp, |||\cdot|||)\| \leqslant \eta_3\sqrt{1+\eta_2}/2$ also. The value η_2 can be chosen as small as we like, so, (52) and (53) can both be achieved. All four equations hold, given suitable growth conditions on **D**. $\square$

Lemma 6.2. *Given growth conditions on* **D**, *the following is true*: *for all $i, j \in \mathbb{N}_0$ with $i < j-1$, we have*

$$\|T_i \bar{Q}_j\| \leqslant \delta_{j-1}^{1/9}. \tag{58}$$

Proof. By (6), for any $\eta \in (0,1)$ we have $\|T_i\bar{Q}_j - T_i P_{i,\eta}\bar{Q}_j\| \leqslant \eta^{1/2}$, so

$$\|T_i \bar{Q}_j\| \leqslant \eta^{1/2} + \|P_{i,\eta}\bar{Q}_j\|.$$

By Lemma 5.4, $\|\bar{Q}_j - Q_j + Q_{j-1}\| \leqslant 5j\delta_{j-1}^{1/8}$. And given $j-1 > i$, two applications of (19) give us $\|P_{i,\eta}(Q_j - Q_{j-1})\| \leqslant \|P_{i,\eta}Q_j - P_{i,\eta}\| + \|P_{i,\eta}Q_{j-1} - P_{i,\eta}\| \leqslant 2(\sqrt{\eta_j} + \delta_j/\eta)^{1/2} + 2(\sqrt{\eta_{j-1}} + \delta_{j-1}/\eta)^{1/2}$. With this in mind, a good choice of η is $\eta = \delta_{j-1}^{1/2}$. Recalling that $\eta_k = \delta_k^{1/2}$, this gives us

$$\|T_i\bar{Q}_j\| \leqslant \delta_{j-1}^{1/4} + 5j\delta_{j-1}^{1/8} + 4\sqrt{\delta_{j-1}^{1/4} + \delta_{j-1}^{1/2}} \leqslant \delta_{j-1}^{1/9}$$

for all $j > 1$, given a growth condition on **D**. Thus the lemma is proved. $\square$

Lemma 6.3. *Given growth conditions on* **D**, *the following is true*: *for all $i, j \in \mathbb{N}_0$ with $i < j-1$, we have*

$$\|T_i^* \bar{Q}_j\| \leqslant \delta_{j-1}^{1/9}, \tag{59}$$

and if $i > j$ we have

$$\|T_i^* Q_j - Q_j\|, \|T_i^*\bar{Q}_j - \bar{Q}_j\| \leqslant 3\delta_i^{1/4}. \tag{60}$$

Proof. This time we use (7) to say, for any $\eta \in (0,1)$ we have $\|T_i^* \bar{Q}_j\| \leqslant \|T_i^* Q_{i,\eta} \bar{Q}_j\| + \eta^{1/2}$, $\leqslant \eta^{1/2} + \|Q_{i,\eta}\bar{Q}_j\| \leqslant \eta^{1/2} + 5j\delta_{j-1}^{1/8} + \|Q_{i,\eta}(Q_j - Q_{j-1})\|$ by Lemma 5.4. And given $j-1 > i$, two applications of (27) give us $\|Q_{i,\eta}(Q_j - Q_{j-1})\| \leqslant \|Q_{i,\eta}Q_j - Q_{i,\eta}\| + \|Q_{i,\eta}Q_{j-1} - Q_{i,\eta}\| \leqslant 3(\sqrt{\eta_j} + \sqrt{\delta_j/\eta})^{1/2} + 3(\sqrt{\eta_{j-1}} + \sqrt{\delta_{j-1}/\eta})^{1/2}$. With this in mind, a good choice of η is $\eta = \delta_{j-1}^{1/2}$. Recalling that $\eta_k = \delta_k^{1/2}$, this gives us

$$\left\|T_i^* \bar{Q}_j\right\| \leqslant \delta_{j-1}^{1/4} + 5j\delta_{j-1}^{1/8} + 6\sqrt{\delta_{j-1}^{1/4} + \delta_{j-1}^{1/4}} \leqslant \delta_{j-1}^{1/9}$$

for all $j > 1$, given a growth condition on **D**. That gives (59); for (60) note that (9) already gives us $\|Q_j T_i^* - Q_j\| \leqslant \delta_i^{1/2}$ when $i > j$; but then if $x \in \operatorname{Im} Q_j$ with $\|x\| = 1$, we have $\|Q_j T_i^* x - x\| \leqslant \delta_i^{1/2}$, in particular $\|Q_j T_i^* x\| \geqslant 1 - \delta_i^{1/2}$, therefore (since $\|T_i\| \leqslant 1$) $\|(I - Q_j)T_i^* x\| \leqslant \sqrt{1 - (1 - \delta_i^{1/2})^2} \leqslant 2\delta_i^{1/4}$. So, $\|T_i^* x - x\| \leqslant \delta_i^{1/2} + 2\delta_i^{1/4} \leqslant 3\delta_i^{1/4}$, that is, $\|T_i^* Q_j - Q_j\| \leqslant 3\delta_i^{1/4}$. Thus the lemma is proved. □

7. Closing arguments, part one: estimating $\|Tx\|$

To conclude our proof we must make two estimates; for all $x \in H$, we have (a) $\|Tx\| \leqslant c|||x|||$ for a suitable c, and (b) $\Re\langle Tx, x\rangle \geqslant c'|||x|||^2$ for a suitable $c' > 0$. For an easy starter, let us estimate $\|Tx\|$, where as usual, $T = \sum_{n=0}^\infty a_i T_i$. Now for $i > j$, we have

$$\|T_i \bar{Q}_j - \bar{Q}_j\| \leqslant \|T_i Q_j - Q_j\| \leqslant \delta_i/\eta_j \leqslant \delta_i^{1/2}, \tag{61}$$

by (9).

Lemma 7.1. *Given growth conditions on* **D**, *the following is true*: *for every* $x \in H$, *we have*

$$\|Tx\| \leqslant \left(1 + \frac{3}{\sqrt{M}}\right) \cdot |||x|||. \tag{62}$$

Proof. $\|Tx\| = \|\sum_{i=0}^\infty a_i T_i x\| = \|\sum_{i=0}^\infty \sum_{j=0}^\infty \bar{Q}_j x\| \leqslant \|\sum_{i=0}^\infty a_i T_i \sum_{j=0}^\infty \bar{Q}_j^2 x\| + \sum_{j=0}^\infty \|(\bar{Q}_j - \bar{Q}_j^2)x\|$ since $\sum_i a_i = 1$. So, pick any $\eta > 0$. Applying Lemma 6.1 with $\eta_3 = \eta/4$, we can estimate the second sum using (56), so

$$\|Tx\| \leqslant \left\|\sum_{i=0}^\infty a_i T_i \sum_{j=0}^\infty \bar{Q}_j^2 x\right\| + \eta/4 \cdot |||x|||. \tag{63}$$

We estimate: for each $i \in \mathbb{N}_0$,

$$\begin{aligned}\left\|T_i \sum_{j=0}^\infty \bar{Q}_j^2 x - \sum_{j=0}^{i-1} \bar{Q}_j^2 x - T_i \bar{Q}_i^2 x - T_i \bar{Q}_{i+1}^2 x\right\| &\leqslant \sum_{j=0}^{i-1} \left\|T_i \bar{Q}_j^2 x - \bar{Q}_j^2 x\right\| + \sum_{j=i+2}^\infty \left\|T_i \bar{Q}_j^2 x\right\| \\ &\leqslant \sum_{j=0}^{i-1} \delta_i^{1/2} \|\bar{Q}_j x\| + \sum_{j=i+2}^\infty \delta_{j-1}^{1/9} \|\bar{Q}_j x\|\end{aligned} \tag{64}$$

by (61) and (58). The last sum is at most $\sum_{j=0}^\infty \delta_{(j-1)\vee 0}^{1/9} \|\bar{Q}_j x\|$. Since $\sum a_i = 1$, we can take a convex combination of these estimates, note that $\sum_{i=0}^\infty a_i \sum_{j=0}^{i-1} \bar{Q}_j^2 x = \sum_{j=0}^\infty s_{i+1} \bar{Q}_j^2 x$, and get

$$\left\|\sum_{i=0}^\infty a_i T_i \sum_{j=0}^\infty \bar{Q}_j^2 x - \sum_{j=0}^\infty s_{j+1} \bar{Q}_j^2 x - \sum_{i=0}^\infty a_i\left(T_i \bar{Q}_i^2 x + T_i \bar{Q}_{i+1}^2 x\right)\right\| \leqslant \sum_{j=0}^\infty \delta_{(j-1)\vee 0}^{1/9} \|\bar{Q}_j x\|.$$

By a familiar estimate, this is no more than $(\sum_{j=0}^\infty \delta_{(j-1)\vee 0}^{1/9})^{1/2} (\sum_{j=0}^\infty \delta_{(j-1)\vee 0}^{1/9} \|\bar{Q}_j x\|^2)^{1/2} \leqslant \eta/4|||x|||$, given a growth condition, such as $\delta_j^{1/9} \leqslant \min(2^{-j-2}, s_{j+1}) \cdot \eta/8$ $(j \in \mathbb{N}_0)$. So,

$$\left\|\sum_{i=0}^\infty a_i T_i \sum_{j=0}^\infty \bar{Q}_j^2 x - \sum_{j=0}^\infty s_{j+1} \bar{Q}_j^2 x - \sum_{i=0}^\infty a_i\left(T_i \bar{Q}_i^2 x + T_i \bar{Q}_{i+1}^2 x\right)\right\| \leqslant \eta/4|||x|||. \tag{65}$$

Next we estimate the terms on the left of (65). $\|\sum_{j=0}^\infty s_{j+1} \bar{Q}_j^2 x\|^2 \leqslant \sum_{j=0}^\infty s_{j+1}^2 \|\bar{Q}_j^2 x\|^2 + \sum_{j\neq k;\, j,k\in\mathbb{N}_0} s_{j+1}s_{k+1}\langle \bar{Q}_j^2 x, \bar{Q}_k^2 x\rangle \leqslant \sum_{j=0}^\infty s_{j+1}\|\bar{Q}_j^2 x\|^2 + \sum_{j\neq k;\, j,k\in\mathbb{N}_0} |\langle \bar{Q}_j^2 x, \bar{Q}_k^2 x\rangle|$, since no s_j exceeds 1. This is $|||x|||^2 + 2\sum_{j>k;\, j,k\in\mathbb{N}_0} |\langle \bar{Q}_j^2 x, \bar{Q}_k^2 x\rangle| \leqslant |||x|||^2 + 2\sum_{j>k} \varepsilon_j' \|\bar{Q}_j x\| \|\bar{Q}_k x\|$, where we define

$$\varepsilon'_j = 5(j+2)\delta_{j-1}^{1/8}, \tag{66}$$

the upper bound for $\|\bar{Q}_j^*\bar{Q}_k\|$ obtained from (41). Given a growth condition, we can assume that (ε'_j) is a decreasing sequence, so for $j > k \geqslant 0$, we have $\varepsilon'_j \leqslant \sqrt{\varepsilon'_j\varepsilon_{k\vee 1}} = \sqrt{\varepsilon_{j\vee 1}\varepsilon_{k\vee 1}}$. So, $\|\sum_{j=0}^\infty s_{j+1}\bar{Q}_j^2 x\|^2 \leqslant |||x|||^2 + \sum_{j=0}^\infty \sqrt{\varepsilon_{j\vee 1}}\|\bar{Q}_j x\| \cdot \sum_{k=0}^\infty \sqrt{\varepsilon_{k\vee 1}}\|\bar{Q}_k x\| = |||x|||^2 + (\sum_{j=0}^\infty \sqrt{\varepsilon_{j\vee 1}}\|\bar{Q}_j x\|)^2$. Splitting up $\sqrt{\varepsilon_{j\vee 1}}\|\bar{Q}_j x\|$ into the product $(\sqrt[4]{\varepsilon_{j\vee 1}})(\sqrt[4]{\varepsilon_{j\vee 1}}\|\bar{Q}_j x\|)$ we may apply Cauchy–Schwartz:

$$\left\|\sum_{j=0}^\infty s_{j+1}\bar{Q}_j^2 x\right\|^2 \leqslant |||x|||^2 + \left(\sum_{j=0}^\infty \sqrt{\varepsilon_{j\vee 1}}\right)\left(\sum_{j=0}^\infty \sqrt{\varepsilon_{j\vee 1}}\|\bar{Q}_j x\|^2\right) \tag{67}$$

$$\leqslant |||x|||^2(1+\eta/4) \tag{68}$$

if we assume as a growth condition that $\sqrt{\varepsilon_{j\vee 1}} \leqslant \min(2^{-j-1}, s_{j+1}) \cdot \sqrt{\eta}/2$ for all $j \in \mathbb{N}_0$ (so the first sum is at most $\sqrt{\eta}/2$, and the second one is at most $\sqrt{\eta}/2 \cdot |||x|||^2$). Returning to (65) we estimate the other terms:

$$\left\|\sum_i a_i T_i \bar{Q}_i^2 x\right\| \leqslant \left(\sum_i a_i\right)^{1/2}\left(\sum_i a_i\|\bar{Q}_x\|^2\right)^{1/2} \leqslant \frac{1}{\sqrt{M}}\left(\sum_i s_{i+1}\|\bar{Q}_x\|^2\right)^{1/2} = \frac{1}{\sqrt{M}}|||x|||, \tag{69}$$

using Cauchy–Schwartz, plus the fact that $\sum a_i = 1$, and $a_I \leqslant s_{i+1}/M$ for all i by (34). Similarly (34) also gives

$$\left\|\sum_i a_i T_i \bar{Q}_{i+1}^2 x\right\| \leqslant |||x||| \cdot \max\left\{\sqrt{\frac{a_i}{s_{i+2}}}\right\} \leqslant \sqrt{\frac{M+2}{M(M+1)}}|||x|||. \tag{70}$$

To conclude the proof, we add up (69), (70) and (68), and substitute in (65): $\|\sum_i a_i T_i \sum_j \bar{Q}_j^2 x\| \leqslant |||x|||(1 + \frac{\eta}{2} + \frac{1}{\sqrt{M}} + \sqrt{\frac{M+2}{M(M+1)}})$, so by (63), $\|Tx\| \leqslant |||x|||(1 + \frac{3\eta}{4} + \frac{1}{\sqrt{M}} + \sqrt{\frac{M+2}{M(M+1)}}) \leqslant |||x|||(1 + \frac{3}{\sqrt{M}})$, if η is chosen suitably small. Thus the lemma is proved. □

8. Closing arguments, part 2: estimating $\Re\langle Tx, x\rangle$

We now begin to estimate $\Re\langle Tx, x\rangle$, where $T = \sum_i a_i T_i$.

Lemma 8.1. *Let $\eta > 0$ be given. Provided* **D** *increases sufficiently rapidly, the following is true. For every $x \in E^\perp$ with $x = \tau y$, we have*

$$\left|\left\langle \sum_{i=0}^\infty a_i T_i x, x\right\rangle - \sum_{\substack{i,j,k\in\mathbb{N}_0 \\ j\leqslant i;\, k\leqslant i+1}} \langle a_i T_i \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle\right| \leqslant \eta |||y|||^2. \tag{71}$$

Proof. $x = \tau y = \sum_j \bar{Q}_j^2 y$ so $\langle \sum_{i=0}^\infty a_i T_i x, x\rangle = \sum_{i,j,k\in\mathbb{N}_0}\langle a_i T_i \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle$. So the error on the left-hand side of (71) is the sum of $\langle a_i T_i \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle$ over values i, j, k such that either $j > i$ or $k > i+1$, or both. So either $k > (i+1)\vee j$ or $j \geqslant (i+1)\vee k$. We use the estimate $\langle T_i\bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle \leqslant \|T_i\bar{Q}_j\|\|\bar{Q}_j y\|\|\bar{Q}_k y\|$ when $j \geqslant (i+1)\vee k$, and the estimate $\langle T_i\bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle \leqslant \|T_i^*\bar{Q}_k\|\|\bar{Q}_j y\|\|\bar{Q}_k y\|$ when $k > (i+1)\vee j$. The left-hand side is at most

$$\sum_{\substack{i,j,k\in\mathbb{N}_0 \\ j\geqslant (i+1)\vee k}} a_i\delta_{j-1}^{1/9}\|\bar{Q}_j y\|\|\bar{Q}_k y\| + \sum_{\substack{i,j,k\in\mathbb{N}_0 \\ k>(i+1)\vee j}} a_i\delta_{k-1}^{1/9}\|\bar{Q}_j y\|\|\bar{Q}_k y\|, \tag{72}$$

because $\|T_i\bar{Q}_j\| \leqslant \delta_{j-1}^{1/9}$ when $j > i$ by Lemma 6.2, and $\|T_i^*\bar{Q}_k\| \leqslant \delta_{k-1}^{1/9}$ when $i < k-1$ by Lemma 6.3. The value of i in a term of the sums in (72) is always at least $j\vee k+1$, and the value of $j-1$ in the first sum ($k-1$ in the second) is never less than zero. So (72) is at most

$$\sum_{j,k=0}^\infty s_{j\vee k+1}\delta_{(j\vee k-1)\vee 0}^{1/9}\|\bar{Q}_j y\|\|\bar{Q}_k y\| \leqslant \left(\sum_{j=0}^\infty \delta_{(j-1)\vee 0}^{1/18}\|\bar{Q}_j y\|\right)^2,$$

since $s_n = \sum_{m=n}^\infty a_m \leqslant 1$ for all n, and $\delta_{(j\vee k-1)\vee 0} \leqslant \sqrt{\delta_{(j-1)\vee 0}\delta_{(k-1)\vee 0}}$ because $(\delta_r)_{r=0}^\infty$ is a decreasing sequence:

$$\leqslant \left(\sum_{j=0}^{\infty} \delta_{(j-1)\vee 0}^{1/18}\right)\left(\sum_{j=0}^{\infty} \delta_{(j-1)\vee 0}^{1/18} \|\bar{Q}_j y\|^2\right)$$
$$\leqslant \eta \|\!|y|\!\|^2 \tag{73}$$

by the Cauchy–Schwartz inequality; and provided we assume, as growth conditions on **D**, that (say) $\delta_{(j-1)\vee 0}^{1/18} \leqslant \eta \cdot 2^{-j-1}$ for all $j \in \mathbb{N}$ (so the first sum is at most η) and also $\delta_{(j-1)\vee 0}^{1/18} \leqslant s_{j+1}$ (so the second sum is at most $\|\!|y|\!\|^2$). Thus the lemma is proved. □

We now split up the second sum in (71) into four parts, and we will estimate each part:

$$\sum_{\substack{i,j,k \in \mathbb{N}_0 \\ j \leqslant i;\, k \leqslant i+1}} \langle a_i T_i \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle = t_1 + t_2 + t_3 + t_4, \tag{74}$$

where

$$t_1 = \sum_i \sum_{j,k<i} a_i \langle T_i \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle, \tag{75}$$

$$t_2 = \sum_i \sum_{j=0}^{i-1} a_i \langle T_i \bar{Q}_j^2 y, \bar{Q}_i^2 y + \bar{Q}_{i+1}^2 y\rangle, \tag{76}$$

$$t_3 = \sum_i \sum_{k=0}^{i-1} a_i \langle T_i \bar{Q}_i^2 y, \bar{Q}_k^2 y\rangle, \tag{77}$$

and

$$t_4 = \sum_{i=0}^{\infty} a_i \langle T_i \bar{Q}_i^2 y, \bar{Q}_i^2 y + \bar{Q}_{i+1}^2 y\rangle. \tag{78}$$

We will estimate each term t_i.

8.1. Estimating the term t_1

By (61), $|t_1 - \sum_{j,k<i} a_i \langle \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle| \leqslant \sum_{j,k<i} a_i \delta_i^{1/2} \|\bar{Q}_j y\| \cdot \|\bar{Q}_k y\| = \sum_{j,k \in \mathbb{N}_0} \|\bar{Q}_j y\| \cdot \|\bar{Q}_k y\| \sum_{i=1+j\vee k}^{\infty} a_i \delta_i^{1/2} \leqslant \sum_{j,k \in \mathbb{N}_0} \varepsilon_j'' \|\bar{Q}_j y\| \varepsilon_k'' \|\bar{Q}_k y\|$ (where $\varepsilon_j'' = (\sum_{i=1+j}^{\infty} a_i \delta_i^{1/2})^{1/2}$) $= (\sum_{j=0}^{\infty} \varepsilon_j'' \|\bar{Q}_j y\|)^2 = (\sum_{j=0}^{\infty} (\sqrt{\varepsilon_j''})(\sqrt{\varepsilon_j''} \|\bar{Q}_j y\|))^2 \leqslant (\sum_{j=0}^{\infty} \sqrt{\varepsilon_j''}) \cdot (\sum_{j=0}^{\infty} \sqrt{\varepsilon_j''} \|\bar{Q}_j y\|^2)$ by the Cauchy–Schwartz inequality. If we assume as growth conditions on **D** that $\sqrt{\varepsilon_j''} \leqslant \min(2^{-j-1}, \eta s_{j+1})$ for $j \in \mathbb{N}_0$, we will have

$$\left| t_1 - \sum_{j,k<i} a_i \langle \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle \right| \leqslant \eta \|\!|y|\!\|^2. \tag{79}$$

8.2. Estimating the term t_2

Using (61) again, we have $|t_2| \leqslant \sum_{j<i} a_i |\langle T_i \bar{Q}_j^2 y, \bar{Q}_i^2 y + \bar{Q}_{i+1}^2 y\rangle| \leqslant \sum_{j<i} a_i \delta_i^{1/2} (\|\bar{Q}_i y\| + \|\bar{Q}_{i+1} y\|) \|\bar{Q}_j y\| + \sum_{j<i} a_i |\langle \bar{Q}_j^2 y, \bar{Q}_i^2 y + \bar{Q}_{i+1}^2 y\rangle|$. Since the sequences (a_j) and (δ_j) are decreasing, we have

$$|t_2| \leqslant \sum_{j<i} a_i^{1/2} \delta_i^{1/4} \big(\|\bar{Q}_i y\| + \big\|\bar{Q}_{i+1}^2 y\big\|\big) a_j^{1/2} \delta_j^{1/4} \|\bar{Q}_j y\| + \sum_{j<i} a_i \big(\varepsilon_i' \|\bar{Q}_i y\| + \varepsilon_i' \|\bar{Q}_{i+1} y\|\big) \|\bar{Q}_j y\| \tag{80}$$

by (41), where ε_i' is as in (66). Given a growth condition we can assume (ε_j') is a decreasing sequence, so the second sum in (80) is at most $(\sum_{i=0}^{\infty} \sqrt{a_i \varepsilon_i'} (\|\bar{Q}_i y\| + \|\bar{Q}_{i+1} y\|))^2 \leqslant (\sum_{i=0}^{\infty} \sqrt{a_i \varepsilon_i'})(\sum_{i=0}^{\infty} \sqrt{a_i \varepsilon_i'} (\|\bar{Q}_i y\| + \|\bar{Q}_{i+1} y\|)^2)$ by Cauchy–Schwartz. This is at most $\eta/4 \|\!|y|\!\|^2$, given a suitable growth condition on **D**. The first term in (80) is at most $(\sum_{i=0}^{\infty} a_i^{1/2} \delta_i^{1/4} (\|\bar{Q}_i y\| + \|\bar{Q}_{i+1} y\|))^2$; by Cauchy–Schwartz this is at most $4(\sum_{i=0}^{\infty} \delta_i^{1/4}) \cdot (\sum_{i=0}^{\infty} \delta_i^{1/4} (\|\bar{Q}_i\| + \|\bar{Q}_{i+1} y\|)^2)$; and given growth conditions such as, e.g., $\delta_i^{1/4} \leqslant \sqrt{\eta} \min(2^{-i-4}, s_{i+2}/8)$, we can be sure that this is at most $\eta \|\!|y|\!\|^2/4$. That gives us a grand total of

$$t_2 \leqslant \eta \|\!|y|\!\|^2/2. \tag{81}$$

8.3. Estimating the term t_3

We have $|t_3| \leqslant |\sum_{k<i} a_i \cdot \langle \bar{Q}_i^2 y, \bar{Q}_k^2 y\rangle| + |\sum_{k<i} a_i \langle \bar{Q}_i^2 y, (T_i^* \bar{Q}_k - \bar{Q}_k)\bar{Q}_k y\rangle| = |\sum_{k<i} a_i \cdot \langle \bar{Q}_i y, (\bar{Q}_i^* \bar{Q}_k)\bar{Q}_k y\rangle| + |\sum_{k<i} a_i \langle \bar{Q}_i^2 y, (T_i^* \bar{Q}_k - \bar{Q}_k)\bar{Q}_k y\rangle|$. From Lemma 6.3 and Lemma 5.4, we deduce that

$$|t_3| \leqslant \sum_{k<i} a_i \|\bar{Q}_i y\| \|\bar{Q}_k y\| \{\varepsilon_i' + 3\delta_i^{1/4}\}. \tag{82}$$

The argument now follows a familiar path: write $\varepsilon_i''' = \varepsilon_i' + 3\delta_i^{1/4}$ and we may assume as a growth condition that ε_i''' is a decreasing sequence. So for $i > k \geqslant 0$ we have $\varepsilon_i''' \geqslant \sqrt{\varepsilon_i''' \varepsilon_{k\vee 1}'''} = \sqrt{\varepsilon_{i\vee 1}''' \varepsilon_{k\vee 1}'''}$. The right-hand side of (82) is at most $\sum_{k<i;i,k\in\mathbb{N}_0} \sqrt{\varepsilon_i'''}\|\bar{Q}_i y\| \cdot \sqrt{\varepsilon_{k\vee 1}'''}\|\bar{Q}_k y\| \leqslant (\sum_{k=0}^\infty \sqrt{\varepsilon_{k\vee 1}'''}\|\bar{Q}_k y\|)^2 \leqslant (\sum_{k=0}^\infty \sqrt{\varepsilon_{k\vee 1}'''})(\sum_{k=0}^\infty \sqrt{\varepsilon_{k\vee 1}'''}\|\bar{Q}_k y\|^2) \leqslant \eta |||y|||^2/4$, using the Cauchy–Schwartz inequality, and assuming growth conditions such as (say) $\sqrt{\varepsilon_k'''} \leqslant \sqrt{\eta}\min(2^{-k-1}, s_{k+1})/2$ (all $k \in \mathbb{N}$). So,

$$|t_3| \leqslant \eta |||y|||^2/4. \tag{83}$$

8.4. Estimating the term t_4

Now $t_4 = |\sum_i a_i(\langle T_i \bar{Q}_i^2 y, \bar{Q}_i^2 y\rangle + \langle T_i \bar{Q}_i^2 y, \bar{Q}_{i+1}^2 y\rangle)| \leqslant \sum_i a_i \|\bar{Q}_i y\|^2 + \sum_i a_i \cdot \frac{1}{2}(\|\bar{Q}_i y\|^2 + \|\bar{Q}_{i+1} y\|^2) = \sum_i a_i \cdot (\frac{3}{2}\|\bar{Q}_i y\|^2 + \frac{1}{2}\|\bar{Q}_{i+1} y\|^2) \leqslant |||y|||^2(\frac{3}{2} \cdot \max\{a_i/s_{i+1}\} + \frac{1}{2} \cdot \max\{a_i/s_{i+2}\}) \leqslant |||y|||^2(\frac{3}{2} \cdot \frac{1}{M} + \frac{1}{2} \cdot \frac{M+2}{M(M+1)})$ by (34); so we may make the simple estimate

$$t_4 \leqslant \frac{2}{M-1}|||y|||^2. \tag{84}$$

8.5. A lower bound for $\Re\langle Tx, x\rangle$

Writing $T = \sum_{i=0}^\infty a_i T_i$, we may substitute our four estimates (79), (81), (83) and (84) in (71), and use the triangle inequality. We obtain

$$\left|\langle Tx, x\rangle - \sum_{j,k<i} a_i \langle \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle\right| \leqslant |||y|||^2\left(\frac{2}{M-1} + \frac{11\eta}{4}\right). \tag{85}$$

Our last estimates involve the sum over $j, k < i$,

$$\sum_{j,k<i} a_i \langle \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle = \sum_{j=0}^\infty s_{j+1}\|\bar{Q}_j^2 y\|^2 + \sum_{j,k\in\mathbb{N}_0, j\neq k} s_{j\vee k+1}\langle \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle = t_5 + t_6, \tag{86}$$

let us say. The term $t_5 = |||y|||^2 - \sum_{j=0}^\infty s_{j+1}(\|\bar{Q}_j y\|^2 - \|\bar{Q}_j^2 y\|^2)$, and the sum on the right is $\sum_{j=0}^\infty s_{j+1}(\|\bar{Q}_j y\| + \|\bar{Q}_j^2 y\|)(\|\bar{Q}_j y\| - \|\bar{Q}_j^2 y\|) \leqslant (\sum_{j=0}^\infty s_{j+1}(\|\bar{Q}_j y\| + \|\bar{Q}_j^2 y\|)^2)^{1/2} \cdot (\sum_{j=0}^\infty s_{j+1}(\|\bar{Q}_j y\| - \|\bar{Q}_j^2 y\|)^2)^{1/2}$ by the Cauchy–Schwartz inequality. By the triangle inequality, we have $(\sum_{j=0}^\infty s_{j+1}(\|\bar{Q}_j y\| + \|\bar{Q}_j^2 y\|)^2)^{1/2} \leqslant (\sum_{j=0}^\infty s_{j+1}\|\bar{Q}_j y\|^2)^{1/2} + (\sum_{j=0}^\infty s_{j+1}\|\bar{Q}_j^2 y\|^2)^{1/2} \leqslant 2|||y|||$; and the second product $(\sum_{j=0}^\infty s_{j+1}(\|\bar{Q}_j y\| - \|\bar{Q}_j^2 y\|)^2)^{1/2} \leqslant (\sum_{j=0}^\infty s_{j+1}\|(\bar{Q}_j - \bar{Q}_j^2)y\|^2)^{1/2} \leqslant \sum_{j=0}^\infty \|(\bar{Q}_j - \bar{Q}_j^2)y\|$, because the l_1 norm exceeds the l_2 norm, and no s_j exceeds 1. By Lemma 5.5, suitable growth conditions on **D** will ensure that this is at most $\sum_{j=0}^\infty \alpha_j \|\bar{Q}_j y\|$ for any sequence (α_j) of positive reals we care to choose; so

$$t_5 \geqslant |||y|||^2 - |||y|||\sum_{n=0}^\infty \alpha_j \|\bar{Q}_j y\| \geqslant |||y|||^2 - 2|||y|||\left(\sum_{n=0}^\infty \alpha_j\right)^{1/2}\left(\sum_{n=0}^\infty \alpha_j \|\bar{Q}_j y\|^2\right)^{1/2} \geqslant |||y|||^2(1-\eta)$$

by the Cauchy–Schwartz inequality, if we choose the α_j so that (say) $\alpha_j \leqslant \eta/2 \cdot \min(2^{-j-1}, s_{j+1})$ (so the first sum will be at most $\eta/2$, and the second at most $\eta/2|||y|||^2$).

The term t_6 in (86) is estimated more simply: $|t_6| \leqslant 2\sum_{j>k}|\langle \bar{Q}_j^2 y, \bar{Q}_k^2 y\rangle| \leqslant 2\sum_{j>k}\|\bar{Q}_j^* \bar{Q}_k\| \cdot \|\bar{Q}_j^* y\|\|\bar{Q}_k^* y\| \leqslant 2\sum_{j>k;j,k\in\mathbb{N}_0} \varepsilon_j' \|\bar{Q}_j^* y\|\|\bar{Q}_k^* y\|$, where ε_j' is as in (66). (67) tells us that, given growth conditions, this is at most $\eta/2 \cdot |||y|||^2$. Therefore $\Re(t_5 + t_6) \geqslant t_5 - |t_6| \geqslant |||y|||^2(1 - 3\eta/2)$, and substituting this estimate back in (86) and (85), we have $\Re\langle Tx, x\rangle \geqslant |||y|||^2(1 - \frac{2}{M-1} - \frac{17\eta}{4})$. Now $y = \tau^{-1}x$, and we know from Lemma 6.1 that given suitable growth conditions we will have $|||I - \tau^{-1}||| \leqslant \eta$. So $|||y||| \geqslant (1-\eta)|||x|||$, and we have $\Re\langle Tx, x\rangle \geqslant |||x|||^2(1 - \frac{2}{M-1} - \frac{17\eta}{4})(1-\eta)^2$. On the other hand, Lemma 7.1 tells us that $\|Tx\| \leqslant (1 + \frac{3}{\sqrt{M}})|||x|||$, so if **D** increases sufficiently rapidly, the operator T satisfies

$$\Re\langle Tx, x\rangle \geqslant \varepsilon\|Tx\|^2, \tag{87}$$

where $\varepsilon = (1 - \frac{2}{M-1} - \frac{17\eta}{4})(1-\eta)^2(1 + \frac{3}{\sqrt{M}})^{-1} > 0$. But now we consider $\|I - \lambda T\|$ for real $\lambda \geqslant 0$:

$$\|(I - \lambda T)x\|^2 = \|x\|^2 + \lambda^2\|Tx\|^2 - 2\lambda\Re\langle Tx, x\rangle \leqslant \|x\|^2 + \|Tx\|^2\lambda(\lambda - 2\varepsilon).$$

If $\lambda \in [0, 2\varepsilon]$ then $\|I - \lambda T\| \leqslant 1$. For the first time, we know that there is a nonzero operator $S \in \mathscr{A}$ with $\|I - S\| \leqslant 1$. Now we can arrange that ε is very close to 1 if we wish (by varying M and the sequence $\mathbf{D}$); so we can have $\lambda \geqslant 2/(1+\zeta)$ for our original ζ given to us at the start of Section 2. If we choose M and $\mathbf{D}$ correctly, there will be an operator T, in the closed convex hull of the operators T_i, $T = \sum_i a_i T_i$ as in Definition 5.1, such that $\|I - 2T/(1+\zeta)\| \leqslant 1$. The operator T satisfies $\|TS_i - S_i\| \vee \|S_i T - S_i\| \leqslant \zeta$, because the T_i all satisfy the estimate (see Definition 2.1), and T belongs to their closed convex hull; so we have found the element $T \in \Gamma$ for which the search began at the start of Section 2. Thus Theorem 1.1 is proved.

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